Introduction to Machine Learning

Kernels and RKHS

1 Kernel of a Sobolev space

We consider the Sobolev space

$$\mathcal{H} = \{f : [0,1] \to \mathbb{R}, \text{ continuous, differentiable a.e., } f' \in L^2([0,1]),\ f(0) = 0\}\ ,$$

endowed with the Hilbert norm

$$|f|_{\mathcal{H}} = \sqrt{\int_0^1 (f')^2}.$$

1. Prove that if $\mathcal{H}$ is a RKHS with kernel k, then for all $x \in [0,1]$ we have

$$f(x) = \int_0^1 f'(y)\, \frac{\partial}{\partial y} k(x,y)\, \mathrm{d}y \quad \text{and} \quad f(x) = \int_0^1 f'(y) \mathbf{1}_{\{y \leq x\}}\, \mathrm{d}y\ .$$

2. What is the reproducing kernel k associated with $\mathcal{H}$?
3. What is the shape of $\phi(x) = k(x, .)$?

2 RKHS associated to the Gaussian kernel

We consider the Gaussian kernel $k_\sigma(x,y) = \exp\left(\frac{-\|x-y\|^2}{2\sigma^2}\right)$. We denote the Fourier transform in $\mathbb{R}^d$ by

$$\mathbf{F}[f](\omega) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-\mathrm{i}\langle \omega, t\rangle} \mathrm{d}t, \quad \text{for } f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \text{ and } \omega \in \mathbb{R}^d\ .$$

The linear span

$$\mathcal{H}_\sigma = \left\{ f \in C_0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \text{ such that } \int_{\mathbb{R}^d} \left|\mathbf{F}[f](\omega)\right|^2 e^{\sigma|\omega|^2/2}\, \mathrm{d}\omega < +\infty \right\}$$

is endowed with the scalar product

$$\langle f, g\rangle_{\mathcal{H}_\sigma} = (2\pi\sigma^2)^{-d/2} \int_{\mathbb{R}^d} \overline{\mathbf{F}[f](\omega)} \mathbf{F}[g](\omega) e^{\sigma|\omega|^2/2}\, \mathrm{d}\omega\ .$$

1. Check that

$$\langle k_\sigma(x, .), f\rangle_{\mathcal{H}_\sigma} = \mathbf{F}^{-1}\big[\mathbf{F}[f]\big](x) = f(x) \quad \forall\ f \in \mathcal{H}_\sigma, \forall\ x \in \mathbb{R}^d.$$

2. What is the RKHS associated to k_σ ?
3. How does $\mathcal{H}_\sigma$ evolve when σ grows ?